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Asymptotic behavior of densities for stochastic functional differential equations *

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Let T and r be positive constants. For $0<\varepsilon\leq 1$ and a deterministic path $\eta\in C([-r,0];\mathbb{R}^{d})$, consider the $\mathbb{R}^{d}$-valued process $X^{\varepsilon}=\{X^{\varepsilon}(t);-r\leq t\leq T\}$ determined by the equation

$$\begin{cases} X^{\varepsilon}(t)=\eta(t) & (-r\leq t\leq 0),\\ \mathrm{d}X^{\varepsilon}(t)=A(X_{t}^{\varepsilon})\,\mathrm{d}t+\varepsilon\sum_{i=1}^{d}B_{i}(X_{t}^{\varepsilon})\,\mathrm{d}W^{i}(t) & (0<t\leq T), \end{cases} \tag{1}$$

where $A, B_{1}, \ldots, B_{d}$ are $\mathbb{R}^{d}$-valued smooth functions on $C([-r,0];\mathbb{R}^{d})$ such that all derivatives of any orders greater than 1 in the Fréchet sense are bounded, $W=\{(W^{1}(t),\ldots,W^{d}(t));0\leq t\leq T\}$ is a d-dimensional Brownian motion starting from the origin, and $X_{t}^{\varepsilon}=\{X^{\varepsilon}(t+u);-r\leq u\leq 0\}$ is the segment of X^{ε}. Such equation is called *the stochastic functional differential equation*, which was first introduced by Itô-Nisio [2]. Since the current state of the solution depends on the past history of the process, the solution X^{ε} is non-Markovian. Under the conditions on the regularity and the boundedness of the coefficients $A, B_{1}, \ldots, B_{d}$, there exists a unique solution (cf. Itô-Nisio [2], Mohammed [5]). Write $X=X^{\varepsilon}|_{\varepsilon=1}$.

Example 1 Consider the case where $d=1$ and $\eta\in C([-r,0];\mathbb{R})$ is deterministic. Let $\rho(\mathrm{d}u)$ be a finite Borel measure on $[-r,0]$, and B be a constant.

$$\begin{cases} X(t)=\eta(t) & (-r\leq t\leq 0),\\ \mathrm{d}X(t)=-\int_{-r}^{0}X(t+u)\,\rho(\mathrm{d}u)\,\mathrm{d}t+B\,\mathrm{d}W(t) & (0<t\leq T). \end{cases}$$

□

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Example 2 Consider the case where $d = 1$ and $\eta \in C([-r,0];\mathbb{R})$ is deterministic. Let A and B be $\mathbb{R}$-valued smooth function on $\mathbb{R}^2$ such that all derivatives of any orders greater than 1 are bounded.

$$\begin{cases} X(t) = \eta(t) & (-r \le t \le 0), \\ \mathrm{d}X(t) = A(X(t), X(t-r))\,\mathrm{d}t + B(X(t), X(t-r))\,\mathrm{d}W(t) & (0 < t \le T). \end{cases}$$

□

Our goals are to study the large deviation principle for the family $\{\mathbb{P} \circ X^{\varepsilon}(t)^{-1}; 0 < \varepsilon \le 1\}$, and the asymptotic behaviour of the density $p^{\varepsilon}(t,y)$ of the probability law of $X^{\varepsilon}(t)$ as $\varepsilon \to 0$. From now on, we shall suppose that the coefficients $B_1, \ldots, B_d$ in the equation (1) satisfy *the uniformly elliptic condition: there exists a positive constant* C_1 *such that*

$$\inf_{v \in \mathbb{S}^{d-1}} \inf_{f \in C([-r,0];\mathbb{R}^d)} \sum_{i=1}^{d} \big(v \cdot B_i(f)\big)^2 \ge C_1. \tag{2}$$

1 Large deviation principle

Denote by $\mathbb{W}_0^d$ the family of $\mathbb{R}^d$-valued continuous functions on $[0,T]$ starting from the origin, and by $\mathbb{H}_0^d$ the subset of $\mathbb{W}_0^d$ such that each component is absolutely continuous, and that the $\mathbb{L}^2([0,T];\mathbb{R}^d)$-norm of the derivative is bounded. For $f \in \mathbb{H}_0^d$, let $Y^f = \{Y^f(t); -r \le t \le T\}$ be the solution to the functional differential equation of the form:

$$\begin{cases} Y^f(t) = \eta(t) & (-r \le t \le 0), \\ \mathrm{d}Y^f(t) = A(Y_t^f)\,\mathrm{d}t + \sum_{i=1}^{d} B_i(Y_t^f)\,\dot{f}(t)\,\mathrm{d}t & (0 < t \le T). \end{cases} \tag{3}$$

Denote by $\mathbb{W}_\eta^d$ the family of $\mathbb{R}^d$-valued continuous functions on $[-r,T]$ with the initial path $\eta \in C([-r,0];\mathbb{R}^d)$, and by $\mathbb{H}_\eta^d$ the subset of $\mathbb{W}_\eta^d$ such that each component is absolutely continuous on $[0,T]$, and that its $\mathbb{L}^2([0,T];\mathbb{R}^d)$-norm of the derivative is bounded. Write $B = (B_1, \ldots, B_d)$. Then, it holds that

Theorem 1 (cf. [3]) *Under the condition* (2) *on the coefficients* $B_1, \ldots, B_d$ *of the equation* (1), *the family of* $\{\mathbb{P} \circ (X^{\varepsilon})^{-1}; 0 < \varepsilon \le 1\}$ *satisfies the large deviation principle with the good rate* *function $\tilde{I}$, where*

$$\tilde{I}(g)=\begin{cases}\dfrac{1}{2}\displaystyle\int_0^T \left|B(g_t)^{-1}\{\dot{g}(t)-A(g_t)\}\right|^2 dt & (g\in\mathbb{H}_\eta^d),\\ +\infty & (g\notin\mathbb{H}_\eta^d).\end{cases} \tag{4}$$

Skethc of the proof. It is well known as the Schilder theorem (cf. Dembo-Zeitouni [1]) that the family $\{\mathbb{P}\circ(\varepsilon W)^{-1}; 0<\varepsilon\le 1\}$ satisfies the large deviation principle with the good rate function I given by

$$I(f)=\begin{cases}\dfrac{1}{2}\displaystyle\int_0^T |\dot{f}(t)|^2 \mathrm{d}t & (f\in\mathbb{H}_0^d),\\ +\infty & (f\notin\mathbb{H}_0^d).\end{cases}$$

At first, we shall consider the case where the $\mathbb{R}^d$-valued functions $B_1,\ldots,B_d$ are bounded. Let $a>0$ and write $\mathbb{H}_{0,a}^d=\{f\in\mathbb{H}_0^d;\ \|\dot{f}\|_{\mathbb{L}^2([0,T];\mathbb{R}^d)}\le a\}$. Then, it can be easily checked via the routine work that the mapping $\Phi_a:\mathbb{H}_{0,a}^d\ni f\longmapsto\Phi_a(f):=Y^f\in\mathbb{W}_\eta^d$ is continuous. Moreover, for any $f\in\mathbb{H}_0^d$ and $\rho>0$, we can find the positive constants α_ρ and ε_ρ such that

$$\begin{aligned}&\mathbb{P}\left[\sup_{-r\le t\le T}|X^\varepsilon(t)-Y^f(t)|>\rho,\ \sup_{0\le t\le T}|\varepsilon W(t)-f(t)|\le\alpha_\rho\right]\\ &\le C_2\exp\left[-C_3\frac{\rho^2}{\varepsilon^2}\right]\end{aligned}$$

for all $0<\varepsilon\le\varepsilon_\rho$, which can be derived by using the martingale representation theorem on stochastic integrals. Hence, the assertion can be obtained from the Schilder theorem stated above, via the argument stated in Dembo-Zeitouni [1].

We shall discuss the general case. Let $R>0$ be sufficiently large, and denote by σ_R the exit time of the process X^ε from the closed ball centered at the origin with the radius R. Write $X^{\varepsilon,R}(t):=X^\varepsilon(t\wedge\sigma_R)$. Then, the Chebyshev type inequality tells us to see that

$$\lim_{R\to+\infty}\limsup_{\varepsilon\searrow 0}\varepsilon\ln\mathbb{P}\left[\sup_{-r\le t\le T}|X^\varepsilon(t)|>R\right]=-\infty,$$

$$\lim_{R\to+\infty}\limsup_{\varepsilon\searrow 0}\varepsilon\ln\mathbb{P}\left[\sup_{-r\le t\le T}|X^\varepsilon(t)-X^{\varepsilon,R}(t)|>\delta\right]=-\infty$$

for any $0<\delta<1$. Since we have already obtained the result on the large deviation principle for the family $\{\mathbb{P}\circ(X^{\varepsilon,R})^{-1}; 0<\varepsilon\le 1\}$ with the good rate function $\tilde{I}_R$, the limiting procedure as $R\to+\infty$ enables us to get the assertion. $\square$

Corollary 1 (cf. [3]) *For each* $0 < t \le T$, *the family* $\{\mathbb{P} \circ X^\varepsilon(t)^{-1}; 0 < \varepsilon \le 1\}$ *satisfies the large deviation principle with the good rate function* $\bar{I}$, *where*

$$\bar{I}(y) = \inf\{\tilde{I}(g); g \in \mathbb{H}^d_\eta, y = g(t)\}. \tag{5}$$

Proof. Since the mapping $\Pi_t : \mathbb{W}_\eta \ni g \longmapsto \Pi_t(g) := g(t) \in \mathbb{R}^d$ is continuous, the assertion is the direct consequence of Theorem 1 and the contraction principle. □

2 Density estimate

At the beginning, we shall apply the Malliavin calculus to the solution process X^ε. Denote by $D = \{D_u; u \in [0,T]\}$ the Malliavin-Shigekawa derivative operator. For each $0 \le t \le T$, successive approximation of the equation (1) tells us to see that $X^\varepsilon(t)$ is smooth in the Malliavin sense. Moreover, for each $0 \le u \le T$, the $\mathbb{R}^d \otimes \mathbb{R}^d$-valued process $\{D_u X^\varepsilon(t); -r \le t \le T\}$ satisfies the equation of the form:

$$\begin{aligned}
D_u X^\varepsilon(t) &= 0 && (-r \le t \le 0 \text{ or } t < u),\\
D_u X^\varepsilon(t) &= \varepsilon \int_0^{u\wedge t} B(X_s^\varepsilon)\,\mathrm{d}s + \int_0^t \nabla A(X_s^\varepsilon) D_u X_s^\varepsilon\,\mathrm{d}s \\
&\quad + \varepsilon \int_0^t \sum_{i=1}^d \nabla B_i(X_s^\varepsilon) D_u X_s^\varepsilon\,\mathrm{d}W^i(s) && (u \le t \le T),
\end{aligned}$$

where ∇ is the Fréchet derivative. For each $s \in [0,T]$, let $Z^\varepsilon(\cdot,s) = \{Z^\varepsilon(t,s); -r \le t \le T\}$ be the $\mathbb{R}^d \otimes \mathbb{R}^d$-valued process determined by the equation

$$\begin{aligned}
Z^\varepsilon(t,s) &= 0 && (-r \le t \le 0 \text{ or } t < s),\\
Z^\varepsilon(t,s) &= I_d + \int_s^t \nabla A(X_u^\varepsilon) Z_u^\varepsilon(\cdot,s)\,\mathrm{d}u \\
&\quad + \int_s^t \sum_{i=1}^d \nabla B_i(X_u^\varepsilon) Z_u^\varepsilon(\cdot,s)\,\mathrm{d}W^i(u) && (s \le t \le T),
\end{aligned}$$

where $Z_u^\varepsilon(\cdot,s) = \{Z^\varepsilon(u+\sigma,s); -r \le \sigma \le 0\}$. Then, we can compute

$$D_u X^\varepsilon(t) = \varepsilon \int_0^{u\wedge t} Z^\varepsilon(t,s) B(X_s^\varepsilon)\,\mathrm{d}s, \tag{6}$$

thus the associated Malliavin covariance matrix $V^\varepsilon(t)$ can be obtained as follows:

$$V^\varepsilon(t) = \int_0^t Z^\varepsilon(t,u) B(X_u^\varepsilon) B(X_u^\varepsilon)^* Z^\varepsilon(t,u)^*\,\mathrm{d}u, \tag{7}$$

where the symbol K^* indicates the transpose of a matrix K. As stated in Kusuoka-Stroock [4], the condition (2) implies that the probability law of $X^\varepsilon(t)$ admits a smooth density $p^\varepsilon(t,y)$ with respect to the Lebesgue measure on $\mathbb{R}^d$.

Applying Corollary 1, the integration by parts formula and the Girsanov transform on W, we can get

Theorem 2 *Under the condition* (2), *it holds that*

$$\lim_{\varepsilon \searrow 0} \varepsilon^2 \ln p^\varepsilon(t,y) = -\bar{I}(y), \tag{8}$$

where $\bar{I}$ is the function introduced in Corollary 1.

Sketch of the proof. We shall prove the upper estimate only. See [3] as for the lower estimate. Let $0 < \sigma < 1$ be sufficiently small, and $\Lambda_\sigma \in C_0^\infty(\mathbb{R}^d; [0,1])$ such that

$$\Lambda_\sigma(z) = \begin{cases} 1 & (|z-y| \leq \sigma), \\ 0 & (|z-y| > 2\sigma). \end{cases}$$

Then, the integration by parts formula leads us to see that

$$p^\varepsilon(t,y) = \mathbb{E}\left[\mathbb{I}_{(y,+\infty)}(X^\varepsilon(t))\,\mathbb{I}_{\mathrm{Supp}[\Lambda_\sigma]}(X^\varepsilon(t))\,\Gamma(X^\varepsilon, \Lambda_\sigma(X^\varepsilon(t)))\right],$$

where $\Gamma(X^\varepsilon, \Lambda_\sigma(X^\varepsilon(t)))$ is the corresponding weight including the Skorokhod integral of $X^\varepsilon(t)$, $DX^\varepsilon(t)$, $\Lambda_\sigma(X^\varepsilon(t))$ and the inverse of $V^\varepsilon(t)$. From Corollary 1, we can get

$$\limsup_{\varepsilon \searrow 0} \varepsilon^2 \ln \mathbb{P}\left[X^\varepsilon(t) \in \mathrm{Supp}[\Lambda_\sigma]\right] \leq -\inf_{y \in \mathrm{Supp}[\Lambda_\sigma]} \bar{I}(y).$$

On the other hand, under the condition (2), we have

$$\mathbb{E}\left[(\det V^\varepsilon(t))^{-p}\right] \leq C_4\, \varepsilon^{-2pd}$$

for any $p > 1$. Taking the limit as $\sigma \searrow 0$ enables us to get the upper estimate. □

3 Remark

Finally, we shall consider the special case:

$$A(f) \equiv 0, \quad B_i(f) = \tilde{B}_i(f(-r), f(0)) \ (i = 1, \ldots, d), \quad \eta(t) = x \ (-r \leq t \leq T),$$

where $\tilde{B}_1, \ldots, \tilde{B}_d$ are the $\mathbb{R}^d$-valued smooth functions on $\mathbb{R}^{2d}$ such that all derivatives of any orders greater than 1 are bounded. Denote by $p(t,y) = p^\varepsilon(t,y)\big|_{\varepsilon=1}$. Then, we have

Theorem 3 *Suppose that the functions $\tilde{B}_1, \ldots, \tilde{B}_d$ satisfy the uniformly elliptic condition: there exists a positive constant C_5 such that*

$$\inf_{v\in\mathbb{S}^{d-1}} \inf_{y,z\in\mathbb{R}^d} \sum_{i=1}^{d} \left(v\cdot\tilde{B}_i(y,z)\right)^2 \geq C_5. \tag{9}$$

Then, for each $0 \leq t \leq T$, the probability law of $X(t)$ has a smooth density $p(t,y)$ such that

$$\lim_{t\searrow 0} t \ln p(t,y) = -r\bar{I}(y), \tag{10}$$

where $\bar{I}$ is the function introduced in Corollary 1.

Sketch of the proof. The existence of the smooth density $p(t,y)$ on the probability law of $X(t)$ can be justified, because of the uniformly elliptic condition (9) on the coefficients $\tilde{B}_1, \ldots, \tilde{B}_d$.

On the other hand, since $X^\varepsilon(t) = x$ for $-r \leq t \leq 0$, we have

$$\begin{aligned} X^\varepsilon(r) &= x + \varepsilon \int_0^r \sum_{i=1}^{d} \tilde{B}_i\big(X^\varepsilon(s-r), X^\varepsilon(s)\big)\, \mathrm{d}W^i(s) \\ &= x + \varepsilon \int_0^r \sum_{i=1}^{d} \tilde{B}_i\big(x, X^\varepsilon(s)\big)\, \mathrm{d}W^i(s). \end{aligned}$$

Similarly, since $X(t) = x$ for $-r \leq t \leq 0$, we see that

$$\begin{aligned} X(\varepsilon^2 r) &= x + \int_0^{\varepsilon^2 r} \sum_{i=1}^{d} \tilde{B}_i\big(X(s-r), X(s)\big)\, \mathrm{d}W^i(s) \\ &= x + \varepsilon \int_0^r \sum_{i=1}^{d} \tilde{B}_i\big(X(\varepsilon^2 s - r), X(\varepsilon^2 s)\big)\, \mathrm{d}\tilde{W}^i(s) \\ &= x + \varepsilon \int_0^r \sum_{i=1}^{d} \tilde{B}_i\big(x, X(\varepsilon^2 s)\big)\, \mathrm{d}\tilde{W}^i(s), \end{aligned}$$

where $\tilde{W} = \{(\tilde{W}^1(t), \ldots, \tilde{W}^d(t))\,;\, 0 \leq t \leq T\}$ is another Brownian motion starting from the origin. In the second equality, we have used the scaling property of Brownian motions. Hence, the uniqueness of the solutions yields that $X(\varepsilon^2 r) = X^\varepsilon(r)$, which implies

$$p(\varepsilon^2 r, y) = p^\varepsilon(r,y).$$

As seen in Section 2, we have already obtained the asymptotic behavior of $p^\varepsilon(r,y)$ as follows:

$$\lim_{\varepsilon\searrow 0} \varepsilon^2 \ln p^\varepsilon(r,y) = -\bar{I}(y).$$

Taking $t = \varepsilon^2 r$ completes the proof. $\square$

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